Theory of Computation
Problem Set 6

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Please start solving these problems immediately, don't procrastinate, and work in study groups.

Please do not simply copy answers that you do not fully understand;

Advice: Please try to solve the easier problems first (where the meta-problem here is to figure out which are the easier ones ☺). Don't spend too long on any single problem without also attempting (in parallel) to solve other problems as well. This way, solutions to the easier problems (at least easier for you) will reveal themselves much sooner (think about this as a "hedging strategy" or "dovetailing strategy").

1. The following problems are from [Sipser, Second Edition]:

   Pages 128-132: 2.4, 2.5, 2.6, 2.9, 2.10, 2.13, 2.16, 2.17, 2.20, 2.21, 2.22, 2.24, 2.27, 2.32, 2.33, 2.36, 2.37, 2.40, 2.41, 2.42, 2.43, 2.44, 2.45

2. Does every context-free language have a proper context-free subset?
   Does every context-free language have a proper context-free superset?

3. Is every subset of a context-free language necessarily context-free?
   Is every superset of a context-free language necessarily non-context-free?

4. Is a countably-infinite union of context-free languages necessarily context-free?
   Is a countably-infinite intersection of context-free languages necessarily context-free?

5. What is the infinite union of all the context-free languages?
   What is the infinite intersection of all the context-free languages?

6. Let YESNO(L)=$\{xy \mid x \in L \text{ and } y \notin L, x,y \in \Sigma^*\}$. Does YESNO preserve context-freeness?

7. Let PALI(L)=$\{w \mid w \in L \text{ and } w^R \in L\}$. Does PALI preserve context-freeness?

8. Given the alphabet $\Sigma=\{a,b,(,),+,*,\emptyset, \varepsilon\}$ construct a context-free grammar that generates all strings in Σ^* that correspond to regular expressions over {a,b}.

9. Construct the smallest possible (in terms of the number of non-terminals and/or production rules) context-free grammar that generates all well-formed parenthesis.

10. Construct a (small) context-free grammar that generates all well-formed nestings of parenthesis ( ) and brackets [ ].

11. Characterize as precisely as you can the class of languages accepted by deterministic pushdown automata with two stacks.

12. Characterize as precisely as you can the class of languages accepted by deterministic pushdown automata with a single "counter" (i.e., stack with only a single-letter stack alphabet).

13. Characterize as precisely as you can the class of languages accepted by deterministic pushdown automata with two "counters" (i.e., two stacks with only a single-letter stack alphabet).

14. Does there exist a context-free grammar for $\{0^i1^j \mid 1 \le i \le j \le 2i\}$?

15. Does there exist a context-free grammar for $\{0^i1^j \mid 0 \le i \le j \le 1.5i\}$?

16. Let $L=\{0^n1^n \mid n\geq 0\}$. Is $\overline{L}$ (i.e. the complement of L) a context-free language?

17. Let $L= \{0^i1^j \mid i \neq j\}$. Is L a context-free language?

18. Is $\{w\in\{a,b\}^* \mid$ w contains an equal number of a's and b's$\}$ a context-free language?

19. We define the SHUFFLE of two strings $v,w \in \Sigma^*$ as:

$$\text{SHUFFLE}(v,w) = \{v_1w_1v_2w_2...v_kw_k \mid v=v_1v_2...v_k,\ w=w_1w_2...w_k,\ \text{and for some } k \geq 1,\ v_i,w_i \in \Sigma^*,\ 1 \leq i \leq k\}$$

For example, 212**ab**1**baa**2**b**22 $\in$ SHUFFLE(**abbaab**,2121222)

Extend the definition of SHUFFLE to two languages $L_1,L_2 \subseteq \Sigma^*$ as follows:

$$\text{SHUFFLE}(L_1,L_2) = \{w \mid w_1\in L_1,\ w_2\in L_2,\ w\in\text{SHUFFLE}(w_1,w_2)\}$$

a) Is the SHUFFLE of two context-free languages necessarily context-free?

b) Is the SHUFFLE of a context-free language with a regular language necessarily context-free?

20. Is $\{v\$w \mid v,w \in \{a,b\}^*, v\neq w\}$ a context-free language?

21. Is $\{vw \mid v,w \in \{a,b\}^*, v\neq w\}$ a context-free language?

22. Is $\{vw \mid v,w \in \{a,b\}^*, v\neq w, |v|=|w|\}$ a context-free language?

23. Determine whether each of the following languages is context-free.

a) $\{a^na^na^n \mid n>0\}$

b) $\{www \mid w \in \{x,y,z\}^*, |w| < 10^{100}\}$

c) $\{vw \mid v,w \in \{a,b\}^*\}$

24. Which of the following modifications / restrictions to PDA's would change the class of languages accepted, relative to "normal" PDA's?

a) The ability to move the read head backwards (as well as forwards) on the input.

b) The ability to write on (as well as read from) the input tape.

c) Having 2 read-heads moving (independently, left-to-right) over the input.

d) Having three stacks instead of one.

e) Having a stack alphabet of at most two symbols.

f) Having a stack alphabet of one symbol.

g) Having a FIFO queue instead of a stack (i.e., write-only at the top of the queue, and read-only at the bottom of the queue).

25. Write a program (in your favorite programming language, e.g. C, C++, Java, Python, etc.) that prints itself out (i.e., prints its own source code exactly, character-for-character, including white spaces), without reading any files, input devices, buffers, Web sites, or any other external input sources. (A blank program isn't a solution here – this is an exercise in self-replication.)

26. Write a self-printing program which is as short as possible (i.e. having the least number of characters in its source code). Also give it the ability to contain arbitrarily large "payload" (say an arbitrary subroutine or code section) that gets copied along with the rest of the source code. Name an application of this capability.